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LETTER TO THE EDITOR

Anticommutator analogue of the Baker–Hausdorff lemma

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Abstract. The anticommutator analogue of the Baker-Hausdorff lemma is formulated and proved by the differential equation method. It is pointed out that this analogue, when suitably transformed, is more convenient for application whenever the operators in question satisfy simpler repeated anticommutator than repeated commutator relations.

The operator identity

$$e^{A}B\,e^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\ldots \tag{1}$$

variously known as the Baker-Hausdorff lemma [1, p 96] or the Lie series [2, 3], is very useful and has numerous applications (see [4-6] for a recent sample). In this letter we formulate a closely related operator identity involving, instead of repeated commutators, the repeated anticommutators, namely

$$e^{A}B\,e^{A}=B+\{A,B\}+\frac{1}{2!}\{A,\{A,B\}\}+\frac{1}{3!}\{A,\{A,\{A,B\}\}\}+\ldots. \tag{2}$$

This operator identity, when suitably transformed, yields for physical applications the more useful identity

$$e^{A}B\,e^{-A}=\left(B+\{A,B\}+\frac{1}{2!}\{A,\{A,B\}\}+\frac{1}{3!}\{A,\{A,\{A,B\}\}\}+\ldots\right)e^{-2A} \tag{3}$$

or equivalently

$$e^{A}\,B\,e^{-A}=e^{2A}\left(B-\{A,B\}+\frac{1}{2!}\{A,\{A,B\}\}-\frac{1}{3!}\{A,\{A,\{A,B\}\}\}+-\ldots\right). \tag{4}$$

The last two operator identities are more convenient than (1) whenever the operators A and B are such that the repeated anticommutators $\hat{A}^{n}_{+}B$ are simpler to evaluate than the corresponding repeated commutators $\hat{A}^{n}_{-}B$. Hereafter we use the notation [6]

$$\hat{A}^{0}_{\pm}B\equiv B \tag{5}$$

$$\hat{A}^{1}_{\pm}B\equiv[A,B]_{\pm}\equiv AB\pm BA \tag{6}$$

and

$$\hat{A}^{n}_{\pm}B\equiv\hat{A}_{\pm}(\hat{A}^{n-1}_{\pm}B)\qquad n=2,3,\ldots. \tag{7}$$

(We use a caret to denote a superoperator.)

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As an elementary example which illustrates this, consider the similarity transformation

$$\sigma_1' \equiv \exp(i\sigma_3\phi/2)\sigma_1 \exp(-i\sigma_3\phi/2). \tag{8}$$

Here, σ_i $(i = 1, 2, 3)$ are the usual 2×2 Pauli spin matrices. Application of the Baker–Hausdorff lemma requires the evaluation of the repeated commutators $\hat{\sigma}^n_{3,-}\sigma_1 (n = 1, 2, \ldots)$ and yields, after some algebra [1, p 159],

$$\sigma_1' = \sigma_1 \cos\phi - \sigma_2 \sin\phi. \tag{9}$$

Application of equation (3) is simpler since σ_3 and σ_1 anticommute

$$\hat{\sigma}^0_{3,+}\sigma_1 = \sigma_1 \tag{10}$$

$$\hat{\sigma}^1_{3,+}\sigma_1 = 0 \tag{11}$$

and consequently

$$\hat{\sigma}^n_{3,+}\sigma_1 = 0 \qquad n = 2, 3, \ldots. \tag{12}$$

Thus (3) gives at once

$$\sigma_1' = \sigma_1 \exp(-i\sigma_3\phi) = \sigma_1(\cos\phi - i\sigma_3 \sin\phi)$$
$$= \sigma_1 \cos\phi - \sigma_2 \sin\phi. \tag{13}$$

We prove (2) (and simultaneously (1)) by the differential equation method [7]. We define the operator

$$L_\pm(\alpha) = e^{\alpha A} B\, e^{\pm\alpha A} \tag{14}$$

dependent on a (continuous) parameter α. Obviously

$$L_\pm(0) = B. \tag{15}$$

Also, since

$$\frac{d}{d\alpha} e^{\pm\alpha A} = \pm A\, e^{\pm\alpha A} = \pm e^{\pm\alpha A} A \tag{16}$$

we have that $L_\pm(\alpha)$ satisfies the following differential equation:

$$\frac{d}{d\alpha} L_\pm(\alpha) = [A, L_\pm(\alpha)]_\pm = \hat{A}_\pm L_\pm(\alpha). \tag{17}$$

On the other hand, we define the operator

$$R_\pm(\alpha) \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \hat{A}^n_\pm B = e^{\alpha\hat{A}_\pm} B \tag{18}$$

which for $\alpha = 0$ reduces to B (see (5))

$$R_\pm(0) = B. \tag{19}$$

Also, using (7),

$$\frac{d}{d\alpha} R_\pm(\alpha) = \sum_{n=1}^{\infty} \alpha^{n-1}_{(n-1)!} \hat{A}^n_\pm B$$
$$= \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} \hat{A}_\pm(\hat{A}^{n-1}_\pm B)$$
$$= \hat{A}_\pm\left(\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} \hat{A}^{n-1}_\pm B\right)$$
$$= \hat{A}_\pm R_\pm(\alpha). \tag{20}$$

Comparing (15) and (17) with (19) and (20) respectively we infer that $L_\pm(\alpha) = R_\pm(\alpha)$ for all α (the operators $L_\pm(\alpha)$ and $R_\pm(\alpha)$ satisfy the same first-order linear differential equation and are equal at $\alpha = 0$). In particular, for $\alpha = 1$ we obtain

$$e^A B\, e^{\pm A} = e^{\hat{A}_\pm} B \tag{21}$$

which proves (1) and (2). Multiplying (2) by e^{-2A} from the right we get (3), while changing $A \to -A$ in (2) and then multiplying by e^{2A} from the left we obtain (4).

A number of other results follow from (21), e.g.

$$e^A\, e^B = (e^{\hat{A}_\pm}\, e^B) e^{\mp A} \tag{22}$$

(the lower sign case being well known [2]), and

$$((e^{\hat{A}_+} B)\, e^{-2A})^n = (e^{\hat{A}_+} B^n)\, e^{-2A} \tag{23}$$

which is analogous to [2]

$$(e^{\hat{A}_-} B)^n = e^{\hat{A}_-} B^n \tag{24}$$

etc. These and other related results will be discussed fully in a future publication.

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